Analysis of a Cournot Duopoly Model's Stability

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***Abstract*:** In this paper, the feedback control methods are applied to a duopoly model based on heterogeneous expectations. This is the time-delayed feedback control of the production system. This control aims to bring this system into instability equilibrium by using delay of state variables. The validity of the control method is proved through theoretical analysis and numerical simulations. Moreover，scope of convergent condition is given. The production model can quickly reach Nash equilibrium after control, providing theoretical reference and production conditions to enterprises.

***Key words*:** dynamical Cournot model, delayed feedback control, the Pareto optimal, Nash equilibrium, game, bounded rationality

1 Introduction

Oligopolistic market is a universal market mechanism, in which a trade is completely controlled by several firms. The firms manufacture the same or homogeneous products and they must consider not only the demand of marker, but also the actions of their competitors [1]. Game theory has been widely applied to oligopolistic markets thank to its ability to consider strategic interactions among firms. Oligopolist is competitive, and the basic solution which refers to competitive equilibrium in Cournot game is Nash equilibrium or Cournot equilibrium. The adjust dynamics to get the Nash equilibrium and the stability are studied by many works [2–9]. But just as what Nash equilibrium reveals, Nash equilibrium reflects individual rationality, but it violates collective rationality – Nash equilibrium of the duopoly game is not Pareto optimal. The prisoners' dilemma shows that, there is a contradiction between individual rationality and collective rationality, and the correct choice based on individual rationality will reduce everybody's welfare. In other words, Pareto improvement cannot be carried on and Pareto optimal cannot be realized by personal interest's maximization. The main question which the prisoners' dilemma poses is whether a cooperative behaviour can emerge among rational and self-interested players whenever there is no formal agreement [10]. In real economical markets we truly can observe that competitors are often able to achieve the cooperation.

Although the duopoly game with output competition (Cournot game) is faced with prisoners' dilemma (Nash equilibrium is not Pareto optimal), it cannot be studied in standard game model with prisoners' dilemma. Because the collection of strategies in this model is a finite set, and in the output competition it is an infinite set. Cafagna [10] has built a strategy with output adjustment (the 'good' strategy), and makes the firms reach a cooperative equilibrium finally. The prisoners' dilemma can be explained based on that. However, the 'good' strategy is based on the premise that producers completely know about their competitors' output and profit. In fact, the producers with mutual competition, or even the producers who have achieved certain cooperation, keep the output, the profit and the related things as the business secrets for their own benefit. So the supposition of incomplete information is more rational. For example, in the model with two producers, as long as one producer does not know the other's cost of production, it is impossible for the first producer to know about the other's profit under different combination of bilateral outputs. That is to say, the first producer cannot have complete information. Then under the premise that each producer incompletely knows about the competitor's information (output, profit and so on), is there a strategy of output adjustment for the producers to use to achieved a cooperative equilibrium?

In this paper, we study that how firms get bigger profits by adjusting their own outputs. It is different from the paper [10-12] that the producers do not know about the market information of the competitor's output and profit, and the cooperative behaviour in duopoly competition is considered with the ‘‘tit-for-tat” conduct.

2. The model

There are two firms produce a homogeneous good in a market .Taking production decisions at discrete time periods $t = 1,2,3,\ldots$. Denoting the quantity of output by each firm at time t is $q_{i,t}\left(i = 1,2\right)$. We have that cost function has the linear form:

$$c_{i,t} = c_i q_{i,t} \quad (1)$$

Let $p(Q)$ denote the inverse demand function:

$$p_t = a - b\sqrt{Q} \quad (2)$$

Where $a,b > 0$, $a > c_i$ and $Q_t = \sum_i q_{i,t}$

Then the profit of player i at time t is given by:

$$\pi_{i,t} = \left(a - b\sqrt{Q} - c_i\right)q_{i,t} \quad (3)$$

This paper is about cooperation under the incomplete information, and the following models are based on the assumption that the firms compare their own profits with the cooperative profit. The solving of the cooperative profit has been introduced in duopoly game theory. The cooperative profit means the profit which is solved by maximizing the sum of all firms' profit. We consider the symmetrical case: $c_1 = c_2 = c$,then can get the cooperative profit, $\pi_c = \dfrac{2(a-c)^3}{27b^2}$,and the cooperative output, $q_c = \dfrac{2(a-c)^2}{9b^2}$

3. The tit-for-tat dynamic strategy

The tit-for-tat strategy is the best behaviour allowing the achievement of cooperation in repeated games [10]. Its characteristic is that every player consists in doing what the opponent did in the previous move. In the paper, we study the Cournot model with the tit-for-tat conduct. And the dynamic equations are based on the incomplete information. Each producer cannot obtain the competitor's complete information, but he completely knows about his own output and profit. The firm i can compare his profit $\pi_{i,t}$ at time t with the cooperative profit π_c which is Pareto optimal. If the cooperative profit $\pi_c - \pi_{i,t} < 0$, then his own profit is more; he extrapolates that the competitor is cooperative, then he will properly reduce his output to continue the cooperation as a ‘‘reward”1; Otherwise, if $\pi_c - \pi_{i,t} > 0$, the firm i cannot realize the cooperative profit, and extrapolates that the competitor is not cooperative, then he will increase his output as ‘‘penalty”.2 For this case ,we get the dynamical systems of q_1 , and q_2 as follows:

$$q_{i,t+1} = q_{i,t} + u_i\left(\pi_c - \pi_{i,t}\right) = q_{i,t} + u_i\left[\pi_c - \left(a - b\sqrt{Q} - c_i\right)\right] \quad (4)$$

where $u_i\left(i = 1,2\right)$ is a adjusting parameter, and $u_i > 0$ .In this model ,Since the firms do not need know the competitor's related information, it is an adjusting strategy with incomplete information. Although its form is simple, it is based on the thoughts of ‘‘tit-for-tat” strategy in prisoners' dilemma game. Now the question is that whether the firms can achieve a cooperative Pareto optimality. With above assumptions, the duopoly game with heterogeneous players is described by a two-dimensional nonlinear map $T\left(q_1(t), q_2(t)\right) \to \left(q_1(t+1), q_2(t+1)\right)$ defined as :

$$T:\begin{cases} q_1(t+1) = q_1(t) + u_1\left[\pi_c - (a-c)q_1(t) + bq_1(t)\sqrt{q_1(t) + q_2(t)}\right] \\ q_2(t+1) = q_2(t) + u_2\left[\pi_c - (a-c)q_2(t) + bq_2(t)\sqrt{q_1(t) + q_2(t)}\right] \end{cases} \quad (5)$$

Where $q_i(t)$ denotes productions of period t , $q_i(t+1)$ represent productions of period $t+1$

In the paper, we are considering an economic model where only nonnegative equilibrium points are meaningful. So we only study the nonnegative fixed points of the map (5), i.e. the solution of the nonlinear algebraic system as:

$$\begin{cases} \pi_c - (a-c)q_1 + bq_1\sqrt{q_1 + q_2} = 0 \\ \pi_c - (a-c)q_2 + bq_2\sqrt{q_1 + q_2} = 0 \end{cases} \quad (6)$$

By setting $q_i(t+1) = q_i(t), i = 1,2$ in system (5), we obtained (6).

Then it is easy to work out an unique fixed point of system (6): $E = \left(q_1^*, q_2^*\right)$,where

$$q_1^* = q_2^* = q_c = \frac{2(a-c)^2}{9b^2}$$

The stability of these equilibriums is based on the eigenvalues of the Jacobian matrix of system (5)

$$J\left(q_1^*,q_2^*\right)=\begin{pmatrix} 1+u_1\left(c-a+\dfrac{b}{2\sqrt{q_1+q_2}}\right) & \dfrac{u_1 b}{2\sqrt{q_1+q_2}} \\ \dfrac{u_2 b}{2\sqrt{q_1+q_2}} & 1+u_2\left(c-a+\dfrac{b}{2\sqrt{q_1+q_2}}\right) \end{pmatrix}$$

We compute the Jacobian matrix J at E then get

$$J\left(q_1^*,q_2^*\right)=\begin{pmatrix} 1-\dfrac{u_1(a-c)}{6} & \dfrac{u_1(a-c)}{6} \\ \dfrac{u_2(a-c)}{6} & 1-\dfrac{u_2(a-c)}{6} \end{pmatrix}$$

By calculation, we get the characteristic polynomial $P(\lambda)$ of the matrix $J\left(q_1^*,q_2^*\right)$ as following:

$$p(\lambda)=\lambda^2-Tr\lambda+Det=0$$

Where Tr is the trace and Det is the determinant of the Jacobian matrix $J\left(q_1^*,q_2^*\right)$.

$$Tr=2-\frac{u_1(a-c)}{6}-\frac{u_2(a-c)}{6}$$

$$Det=1-\frac{u_1(a-c)}{6}-\frac{u_2(a-c)}{6}$$

Then we have two eigenvalues of matrix $J\left(q_1^*,q_2^*\right)$, $\lambda_1=1$ and $\lambda_2=1-\dfrac{(u_1+u_2)(a-c)}{6}$. If it holds that $u_i\,(i=1,2)$ is very small, we have $|\lambda_2|<1$. Since $\lambda_1=1$ is a critical condition we cannot know the stability of the system (5). But the following numerical

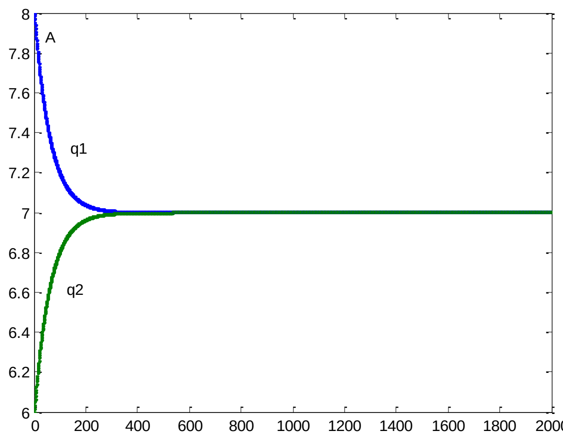

Fig. 1a. The output of the system (4) is stable

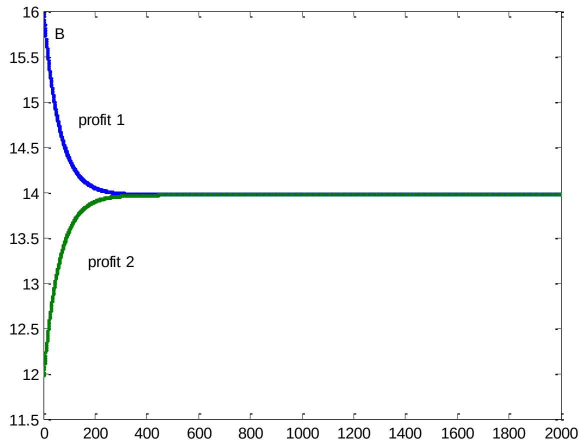

Fig. 1b. The profit of the system (4) is stable

experiments show that its stability is sensitive to the parameter

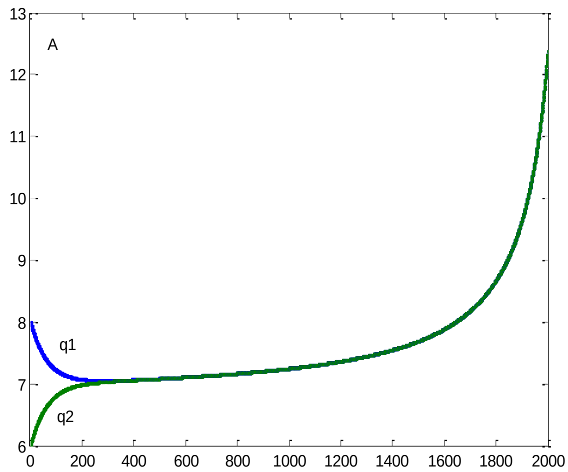

Fig. 2a The output t of the system (4) is unstable

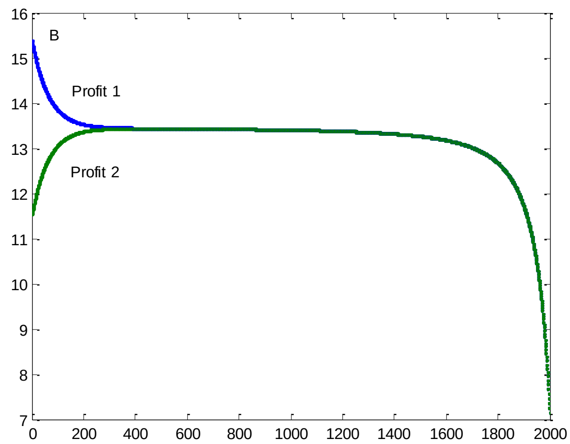

Fig. 2b. The profit of the system (4) is unstable

We take $a = 8, c = 2, u_1 = u_2 = 0.0082$, and the initial value $q_{1,0} = 8, q_{2,0} = 6$. If we fix other parameters and vary one, for instance b, the stability of system changes . Fig. 1 shows that it is stable, but if a parameter changes slightly, it is the contrary (Fig. 2). And Fig. 2 shows that the output and the profit not only cannot achieve the Pareto optimality, but also appears the phenomenon of malignant competition – the outputs of both sides increase infinitely (Fig. 2A), while the profits approach to zero (Fig. 2B). That is to say, the firms in Cournot game cannot achieve the Pareto optimal equilibrium under the adjustment Eq. (4)

4. Delayed feedback control of the production system

4.1 By adding a time-delayed feedback control, we consider a new strategy:

$$\begin{cases} q_{1,t+1} = q_{1,t} + u_1\left(p_c - (a-c)q_{1,t} + b\sqrt{q_{1,t}+q_{2,t}}\right) + k\left(q_{1,t} - q_{1,t-1}\right) \\ q_{2,t+1} = q_{2,t} + u_2\left(p_c - (a-c)q_{2,t} + b\sqrt{q_{1,t}+q_{2,t}}\right) \end{cases} \tag{7}$$

Where $u_i\,(i = 1,2)$ is an adjustment parameter , and $u_i > 0$, $k\left(q_{1,t} - q_{1,t-1}\right)$ is the delayed feedback control of the system. (7) equivalent to the following three-dimensional equations:

$$\begin{cases} q_{1,t+1} = q_{1,t} + u_1\left(p_c - (a-c)q_{1,t} + b\sqrt{q_{1,t}+q_{2,t}}\right) + k\left(q_{1,t} - q_{1,t-1}\right) \\ q_{2,t+1} = q_{2,t} + u_2\left(p_c - (a-c)q_{2,t} + b\sqrt{q_{1,t}+q_{2,t}}\right) \\ q_{3,t+1} = q_{1,t} \end{cases} \tag{8}$$

The Jacobian matrix at $E^* = \left(q_1^*, q_2^*\right)$ takes the form:

$$J(E^*) = \begin{pmatrix} k + 1 + u_1\left(c - a + \dfrac{b}{2\sqrt{q_1^* + q_2^*}}\right) & \dfrac{u_1 b}{2\sqrt{q_1^* + q_2^*}} & -k \\ \dfrac{u_2 b}{2\sqrt{q_1^* + q_2^*}} & 1 + u_2\left(c - a + \dfrac{b}{2\sqrt{q_1^* + q_2^*}}\right) & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

By calculation, we get the characteristic polynomial $f(\lambda)$ of the matrix $J\left(q_1^*, q_2^*\right)$ as following:

$$f(\lambda) = \lambda^3 + B_1\lambda^2 + B_2\lambda + B_3 = 0$$

Where $B_1 = -k - 2 + (u_1 + u_2)\left(a - c - \dfrac{3b^2}{4(a-c)}\right)$

$$B_2 = 2k + 1 + \left(u_1 + u_2 + u_2 k + u_1 u_2\left(-a + c + \frac{3b^2}{4(a-c)}\right)\right)$$
$$*\left(-a + c + \frac{3b^2}{4(a-c)}\right) - \frac{9u_1u_2b^4}{16(a-c)^2}$$
$$B_3 = -k + u_2 k\left(a - c - \frac{3b^2}{4(a-c)}\right)$$

From Jury conditions, the necessary and sufficient conditions for $|\lambda_i| < 1, i = 1,2,3$ are:

$$\begin{cases} 1 + B_1 + B_2 + B_3 > 0 \\ 1 - B_1 + B_2 - B_3 > 0 \\ 1 - B_3^2 > |B_2 - B_1B_3| \\ |B_3| < 1 \end{cases} \tag{9}$$

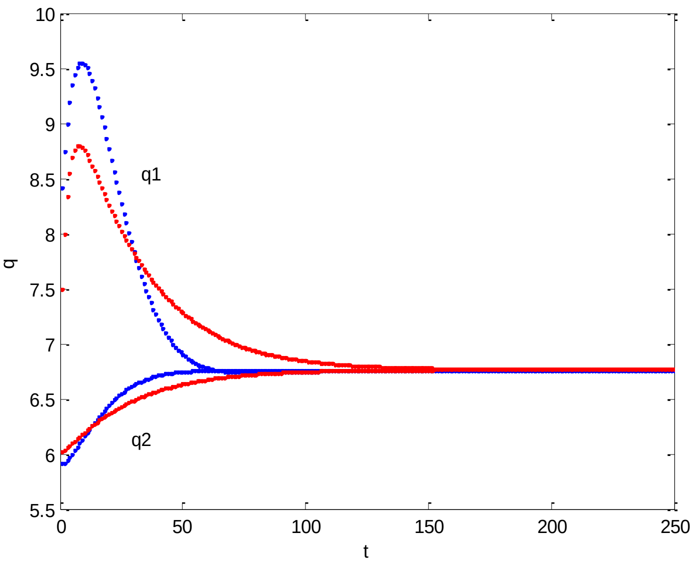

Fig3a

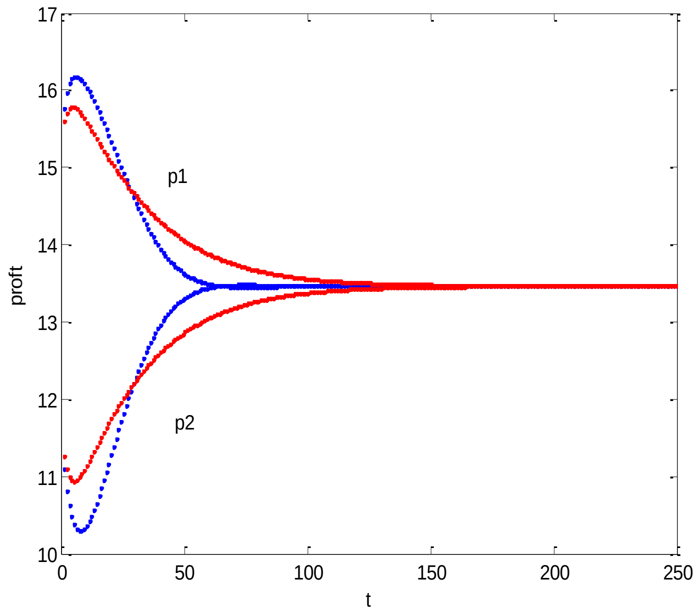

Fig3b

So the equilibrium point E^* of the system (7) is stable, if the conditions in (9) are all satisfied.

We reconsider the unstable situation ($a=8, b=1.09, c=2, u_1=u_2=0.0082, q_{1,0}=8, q_{2,0}=6$) in Section 3. Let $k=0.4$, now the output and profit system (7) become stable, as showed in Fig. 3(blue line) . In Fig.3a,the blue point shows that the changes of Productions1,2,when adds a time-delayed feedback control strategy. In Fig.3b,the blue point shows the proft.

4.2 By adding two time-delayed feedback control, we consider a new strategy:

$$\begin{cases} q_{1,t+1} = q_{1,t} + u_1(p_c - (a-c)q_{1,t} + bq_{1,t}\sqrt{q_{1,t}+q_{2,t}}) + k_1(q_{1,t} - q_{1,t-1}) \\ q_{2,t+1} = q_{2,t} + u_2(p_c - (a-c)q_{2,t} + bq_{2,t}\sqrt{q_{1,t}+q_{2,t}}) + k_2(q_{2,t} - q_{2,t-1}) \end{cases}$$

(10)

Where $u_i\,(i=1,2)$ is an adjustment parameter , and $u_i > 0, k_i(q_{1,t} - q_{1,t-1})$ is the delayed feedback control of the system. (10) equivalent to the following four-dimensional equations:

$$\begin{cases} q_{1,t+1} = q_{1,t} + u_1(p_c - (a-c)q_{1,t} + bq_{1,t}\sqrt{q_{1,t}+q_{2,t}}) + k_1(q_{1,t} - q_{3,t}) \\ q_{2,t+1} = q_{2,t} + u_2(p_c - (a-c)q_{2,t} + bq_{2,t}\sqrt{q_{1,t}+q_{2,t}}) + k_2(q_{2,t} - q_{4,t}) \\ q_{3,t+1} = q_{1,t} \\ q_{4,t+1} = q_{2,t} \end{cases}$$

(11)

The Jacobian matrix at $E^* = (q_1^*, q_2^*)$ takes the form :

$$J(E) = \begin{pmatrix} M_1 & \dfrac{bq_1^*}{2\sqrt{q_1^*+q_2^*}} & -k_1 & 0 \\ \dfrac{bq_2^*}{2\sqrt{q_1^*+q_2^*}} & M_2 & 0 & -k_2 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

By calculation, we get the characteristic polynomial $f(\lambda)$ of the matrix $J(q_1^*, q_2^*)$ as following:

$$f(\lambda) = \lambda^4 + (M_1 + M_2)\lambda^3 + (M_1M_2 + k_2)\lambda^2 + \left[k_1 - k_2M_1 - \frac{b^2q_1^*q_2^*}{4(q_1^*+q_2^*)}\right]\lambda + (k_1k_2 - k_1M_2)$$

Where

$$M_1 = k_1 + 1 - u_1(a-c) + u_1b\left(\frac{3q_1^* + 2q_2^*}{2\sqrt{q_1^*+q_2^*}} + q_1^*\right)$$

$$M_2 = k_2 + 1 - u_2(a-c) + u_2b\left(\frac{2q_1^* + 3q_2^*}{2\sqrt{q_1^*+q_2^*}} + q_2^*\right)$$

$$q_1^* = q_2^* = \frac{2(a-c)^2}{9b^2}$$

From Jury conditions, the necessary and sufficient conditions for $|l_i| < 1, i = 1,2,3,4$ are:

$$\begin{cases} 1 + B_1 + B_2 + B_3 + B_4 > 0 \\ 1 - B_1 - B_2 - B_3 + B_4 > 0 \\ |B_4| < 1 \\ |A_4| < |A_1| \\ |C_1| < |C_3| \end{cases} \qquad (12)$$

Where $B_1 = M_1 + M_2$, $B_2 = M_1M_2 + k_2$, $B_3 = k_1 - k_2M_1 - \frac{b^2 q_{1,t}q_{2,t}}{4(q_{1,t} + q_{2,t})}$, $B_4 = k_1k_2 - k_1M_2$; $A_1 = 1 - B_1^2$, $A_2 = B_1 - B_3B_4$, $A_3 = B_2 - B_2B_4$, $A_4 = B_3 - B_1B_4$, $C_1 = A_4^2 - A_1^2$, $C_2 = A_4A_3 - A_1A_2$, $C_3 = A_4A_2 - A_1A_3$.

So the equilibrium point E^* of the system (10) is stable, if the conditions in (12) are all satisfied.

We reconsider the unstable situation ($a = 8, b = 1.09, c = 2, u_1 = u_2 = 0.0082, q_{1,0} = 8, q_{2,0} = 6$) in Section 3. Because it(12) is so hard to solve, we give the control results $k_1 = 0.85$, and $k_2 = 0.2$. Let $k_1 = 0.85$, and $k_2 = 0.2$. Now the output and profit systems (10) become stable, as Fig. 3 shows. In Fig.3a,the red point shows that the changes of Productions1,2,when both manufacturers introduce time-delayed feedback control strategy. In Fig.3b,the blue point shows the proft.

Through numerical simulation，we obtained the results (Fig 3). This two methods can also control unstable system to achieve stable state. The stability of two strategies is at the same point ; It also shows that two manufacturers introducing time-delay feedback will make the early production more violently shocks . If the control is bad, it may lead the field production into chaos, thereby affecting the market stability and economic benefits. However, by introducing two time-delay feedbacks will make system achieve stability in shorter time than by adding one.

5. Conclusion

This article is about Cournot game for the competition of output. And we have studied two strategies of output's adjustment under incomplete information – the tit-for-tat strategy with time-delayed feedback. In conclusion, the cooperation may be achieved under the tit-for-tat strategy. But the stability of the adjustment system is sensitive to the parameters, and the Pareto Optimality cannot be assured. By introducing the feedback control to the cooperation intention of the players, the firms' cooperation can be achieved, and the Pareto Optimality is stable within the parameters' certain field. So the cooperation can be the result of such a strategy under the certain condition.

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