A Study on Adjacency Matrix for Zero-Divisor Graphs over Finite Ring of Gaussian Integer

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ABSTRACT : The paper studies the characterization of adjacency matrix corresponding to zero-divisor graphs of finite commutative ring of Gaussian integer under modulo 'n'. For each positive integer we calculate number of zero-divisors & examine nature of the matrix, and then we generalized the order of matrix in each case. Firstly, we have started with some example, which motivates the later results. The study is useful in computer science application such as: coding theory, network communication, museum guard problems, etc.

Keywords: Gaussian Integer, Zero-divisor, Adjacency Matrix, Commutative ring.

AMS Classification: 05Cxx; 14C20; 15Axx,13Axx.

1. INTRODUCTION

Let R be finite commutative ring of Gaussian integer, Gaussian integer contains set of all complex numbers a+ib, where a and b are integer. It is denoted by Z[i]. It forms Euclidian domain under usual complex operations, with norm $N(a+ib) = a^2+b^2$. It is clear that a+ib is unit in Z[i] iff N(a+ib) =1, which implies that 1, -1, i, -i are only units.

Let <n> be the principal ideal generated by n in Z[i], where n is a natural number and let Z_n={0,1,2,3,4….n-1} be ring of integer modulo n. The factor ring Z[i]/<n> is isomorphic to $Z_n[i]$ = {a+ib : a,b Z_n} which implies that $Z_n[i]$ is Principal ideal ring. Therefore $Z_n[i]$ is ring of Gaussian integer under modulo n. Consider $Z_n[i]$, and let Z(R) be set of zero divisors of $Z_n[i]$ and G be zero divisor graph of $Z_n[i]$. Consider a, b $Z_n[i]$, then a and b are said to be adjacent if a.b = b.a = 0. The ring of Gaussian integer $Z_p[i]$ forms field if p≡3(mod4) [1], therefore the graph G has no edge if p≡3(mod4).

The concept of zero divisors graph was given by I. Beck [2] but his motive was in coloring of graphs. In [2] Anderson and Livingston associate to a commutative ring with unity a graph ΓR , whose vertex was $Z(R)^{\bullet}$ = Z(R) –{0}. The Zero divisor graphs also have discussed and studied for semi groups by De Meyer [4]. Redmond has generalized the notation of zero divisor graphs. On studying this article it is found that now considerable work has been done in this direction. Some time the zero divisor graph for R is allowed to have '0' as a vertex, in such case '0' has an edge to every other vertex in graphs. For simplification, we have used the definition excluding '0' as a vertex. In the first instance of this paper we consider some finite rings of Gaussian integer and discuses nature of adjacency matrix in each case depending on 'n' we also investigate that what can be the order of matrix in each case, we have started with some example which illustrates the general results. The definition of adjacency matrix for zero divisor graphs is as follows:

$$d_{ij} = \begin{cases} 1, & \text{If } v_i \ \& \ v_j \text{ represent zero} - \text{divisor} \\ 0, & \text{Otherwise} \end{cases},$$

where v_i and v_j are vertices of graph G.

One more advantage of the graph that it also detects the nilpotent element of index 2, when self loop found. We will take basic definition from graph theory [5]-[6] for commutative ring with unity [7]. To avoid trivialities when G has no edges, we will assume when necessary R is not Integral domain, (i.e., we left case p≡3(mod4) [4]). We study for the following rings $Z_p[i]$, $Z_{p^n}[i]$, $Z_{pq}[i]$. Some examples are giving below for each case of $Z_p[i]$:

I. p≡2 (mod4)
II. p≡1 (mod4)

And for ring $Z_{p^n}[i]$, n>1 the cases are such as:

I. p≡1(mod 4)
II. p≡3(mod 4)

III. p≡2(mod 4) and

Last case for $Z_n[i]$, when n = p.q, n≡2(mod 4)

2. The ring $Z_p[i]$,

Case 2.1: When p≡2(mod4), i.e., R = $Z_2[i]$

Firstly we move to discuss about R = $Z_2[i]$

The set of zero divisor Z(R) = {1+i}, the possible edge is self loop.

Graph for Z(R) is given as:

Fig. 1: Zero divisor graph of the $Z_2[i]$

and the adjacency matrix is given as:

$$[1]_{1\cdot 1}$$

Observation from matrix:

i. Ring has only one zero divisor

ii. Matrix is of order '1' and having trace and determinant equal.

Case 2.2: When p≡1(mod4), i.e., R = $Z_5[i]$ and R=$Z_{13}[i]$.

Case 2.2.1: For R = $Z_5[i]$.

The set of zero divisor Z(R) = {2+i, 3+i, 4+2i, 2+4i, 1+3i, 1+2i, 4+3i, 3+4i}

The graph is shown below:

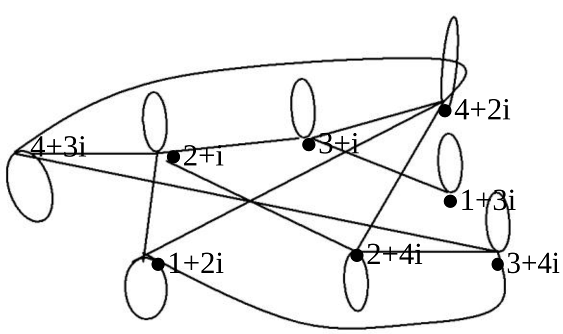


Fig. 2: Zero divisor graph of the $Z_5[i]$

The matrix corresponding to zero divisor graph of $Z_5[i]$

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}_{8\cdot 8}$$

Observation from matrix:

i. The matrix corresponding to zero divisor graph of $Z_5[i]$ is non singular.

ii. All vertices have self loop so trace of the matrix is 8.

iii. The rank of the matrix is 8, therefore zero is not the eigen value of the above matrix.

Case 2.2.2: For R = $Z_{13}[i]$.

The set of zero divisor = {1+5i, 1+8i, 2+3i, 2+10i, 3+2i, 3+11i, 4+6i, 4+7i , 5+12i, 6+4i, 6+9i, 7+4i, 7+9i, 8+i, 8+12i, 9+6i, 9+7i, 10+2i, 10+11i, 11+3i, 11+10i, 12+5i, 12+8i}

The graph is shown below:

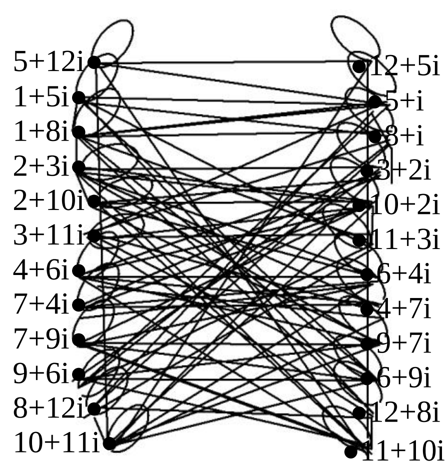


Fig. 3: Zero divisor graph of the $Z_{13}[i]$

The adjacency matrix will be of order 24×24 and can be constructed by using definition.

Observation from matrix:

i. The matrix corresponding to zero divisor graph of $Z_5[i]$ is non singular.

ii. Trace of the matrix will be 24.

iii. Zero will not be the eigenvalue of matrix as rank of adjacency matrix is 24.

Theorem 2.1: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_p[i]$, $p\equiv 1(mod4)$, p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_p[i]$. The order of adjacency matrix in such case is always 2(p-1) ×2(p-1).*

Proof: The ring of Gaussian integer Zn[i] is finite commutative ring under modulo 'n'. It is known by theorem [1] that in a finite commutative ring each non-zero element is either a unit or a zero divisor. To obtain number of zero divisor we subtract units from non zero element. It is found that units in $Z_p[i]$ are φ(p) × φ(p), therefore no. of zero divisor = p^2-(p-1) × (p-1)-1 = 2(p-1)

Thus it have proved that order of matrix is 2(p-1) × 2(p-1).

Theorem 2.2: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_p[i]$, $p\equiv 1(mod4)$, p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_p[i]$. Then adjacency matrix will always be non singular.*

Proof: Let us consider $Z_p[i]$, $p\equiv 1(\text{mod } 4)$ as above discussed example it has observed that if a+ib represent zero divisor then b+ia also represent zero divisor. From the graph it is found there is no such vertex at which a+ib and b+ia both are connected, thus in adjacency matrix no two rows are identical. Therefore we have shown that matrix will be non singular.

Theorem 2.3: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_p[i]$, $p\equiv 1(mod4)$, p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_p[i]$. Then trace of adjacency matrix is always equal to number of zero divisor.*

Proof: Let us consider $Z_p[i]$, $p\equiv 1(\text{mod } 4)$ as above discussed example it has observed that if a+ib represent zero divisor then b+ia also represent zero divisor. From the graph it is found that all the vertices has self loop, therefore trace is equal to number of zero divisor.

3. The ring $Z_{p^n}[i]$

Case 3.1: When $p\equiv 3(mod4)$, i.e., R = $Z_9[i]$, $Z_{27}[i]$

Let R = $Z_9[i]$, the set of the zero divisor is Z(R) = {3, 6, 3i, 6i, 3+3i, 6+3i, 6+6i, 3+6i}

The corresponding zero divisor graph is represented as:

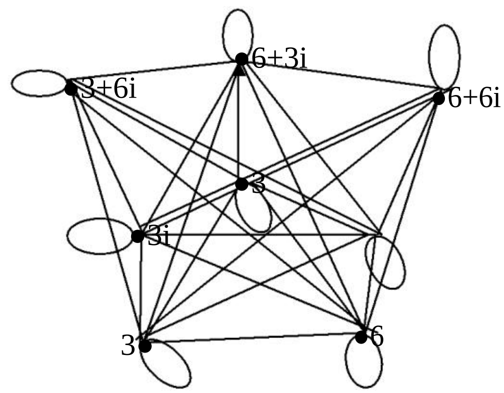


Fig. 4: Zero divisor graph of the $Z_9[i]$

Matrix for the above figure is

$$\begin{bmatrix} 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1\\ 1&1&1&1&1&1&1&1 \end{bmatrix}_{8\cdot 8}$$

Observation from matrix:

i. The matrix for the zero divisor of $Z_9[i]$ is singular.

ii. Trace of the adjacency matrix is 8, as all the vertices have self loop.

iii. Rank of the matrix is 1.

iv. Eigen values of above matrix are 0 and 8.

v. Adjacency matrix corresponding to Z(R) is diagonalizable.

Theorem 3.1: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_{p^n}[i]$, $p\equiv 3(mod4)$, p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_{p^n}[i]$, the order of adjacency matrix in such case is always* $p^{2n}-8p^{2n-2}-1\times p^{2n}-8p^{2n-2}-1$.

Proof: **S**imilar as theorem 1.

Theorem 3.2: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_{p^n}[i]$, $p \equiv 3(mod4)$, p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_{p^n}[i]$. Then trace of adjacency matrix is always natural number n>1.*

Proof: Let us consider $Z_{p^n}[i]$, $p \equiv 3$(mod 4), here p, p^2, p^3… or p^{n-1} as well as ip, ip^2, ip^{n-1} represent zero divisor with itself, i.e., graph must have self loop at least two pairs which are conjugate. Therefore, at least the diagonal entry, i.e., a_{ii} and a_{jj} of the adjacency matrix contain 1. Thus the trace of matrix is natural number 'n', n>1.

Case 3.2: When $p \equiv 2$(mod4), i.e., R = Z_4[i], Z_8[i]

Case 3.2.1 for R = Z_4[i], the set of zero divisors for Z_4[i] = {2, 2i, 1+i, 3+i, 2+2i, 1+3i, 3+3i}

The possible edges for the graph are {2,2}, {2i,2i}, {2+2i,2+2i}, {2+2i,1+i}, {2+2i,3+i}, {2,2i}, {2,2+2i}, {2i, 2+2i}, {3+3i,2+2i}, {2+2i,1+3i}

Graph is given as

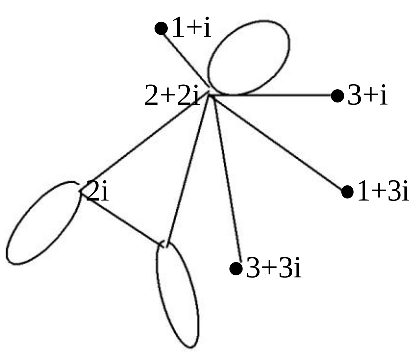


Fig. 5: Zero divisor graph of the Z_4[i]

The adjacency matrix for the graph:

1	1	1	0	0	0	0
1	1	1	1	1	1	1
1	1	1	0	0	0	0
0	1	0	0	0	0	0
0	1	0	0	0	0	0
0	1	0	0	0	0	0
0	1	0	0	0	0	0

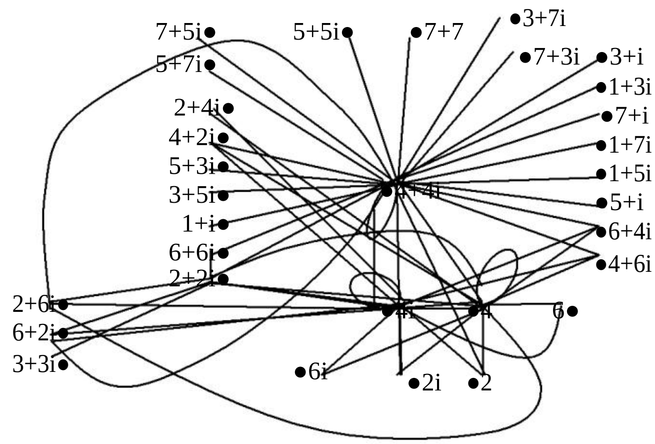
$_{7 \cdot 7}$

Observation from matrix:

i. The Adjacency matrix with respect to zero divisor graph of Z_4[i] is singular.
ii. The trace of above matrix is 3.
iii. Rank of adjacency matrix is 3, which is less than seven so zero must be eigenvalue.

Case 3.2.2 for R = Z_8[i]

The set of zero divisor Z(R) = {2, 4, 2i, 4i, 6, 6i, 1+i, 4+4i,



1+5i, 1+7i, 5+i, 3+i, 2+2i, 3+3i, 2+6i, 6+2i, 7+7i, 4+6i, 6+4i, 7+i, 5+7i, 7+5i, 5+5i, 4+2i, 2+4i, 3+5i, 5+3i, 4+4i, 1+3i, 3+7i, 7+3i} and graph of the zero divisor is given as:

Fig.6: Zero divisor graph of the Z_8[i]

and the matrix for the zero divisor graph will be of order 31×31 and can be constructed in similar manner.

Observation from matrix:

i. The Adjacency matrix with respect to zero divisor graph of Z_8[i] is singular.
ii. The trace of above matrix is 3.
iii. Rank of adjacency matrix is 10.

Theorem 3.3: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_{p^n}[i]$, $p \equiv 2$ (mod4), p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_{p^n}[i]$. The order of adjacency matrix in such case is always $\frac{1}{2}p\,2^{n-1} \cdot \frac{1}{2}p^{2n-1}$.*

Proof: Similar as theorem 1 as the units of $Z_{p^n}[i]$, are $\frac{1}{2}p^{2n}$ by using the reference [1].

Theorem 3.4: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_{p^n}[i]$, p≡2(mod4), p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_{p^n}[i]$. The adjacency matrix in this case is always singular.*

Proof: Let us consider $Z_{p^n}[i]$, p≡2(mod4) as above discussed example it has observed that if a+ib represent zero divisor then b+ia also represent zero divisor. From the graph, it is found that at least two vertices $(p^n-1)(1+i)$ and (1+i) represent zero divisor with $\frac{1}{2}p^n\cdot(1+i)$. In adjacency matrix at least two rows will be identical, thus determinate of matrix is zero.

Theorem 3.5: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_{p^n}[i]$, p≡2(mod4) p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_{p^n}[i]$. Then trace of matrix in such case is always three.*

Proof: Let us consider $Z_{p^n}[i]$, p≡2(mod4) as above discussed example it has observed that if a+ib represent zero divisor then b+ia also represent zero divisor. From the graph it is found that vertices $\frac{i}{2}p^n$, $\frac{1}{2}p^n$ and $\frac{1}{2}p^n\cdot(1+i)$ which produces self loop always. Thus trace of adjacency matrix is three [1].

4. The ring $Z_n[i]$

When n = p.q, n≡2(mod4), where p and q are distinct prime numbers, i.e., R = $Z_6[i]$, $Z_{10}[i]$,

Case 4.1 for R = $Z_6[i]$,

Consider $Z_6[i]$, the set of zero divisors Z(R)= {2, 3, 4, 2i, 3i, 4i, 3+3i, 1+i, 2+2i, 4+4i, 2+4i, 3+i, 5+i, 1+3i, 5+3i, 5+5i, 3+5i, 4+2i, 1+5i}.

The zero divisor graph is shown as:

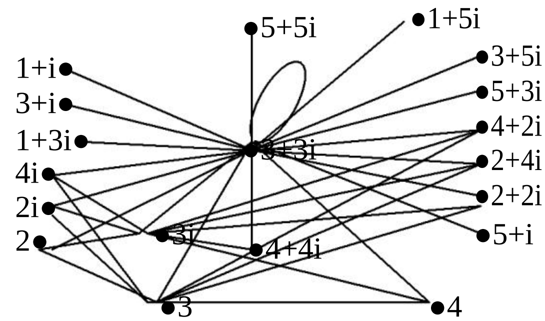


Fig. 7: Zero divisor graph of the $Z_6[i]$

The adjacency matrix is of order 19×19 and can be constructed similarly.

Observation from matrix:

i. The determinant of the matrix is zero.
ii. Trace of the adjacency matrix is 1.
iii. Rank of the matrix is 4.
iv. Matrix is singular so, zero must be eigen value.

Theorem 4.1: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_{p^n}[i]$, p≡2(mod4), p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_{p^n}[i]$. The adjacency matrix in this case is always singular.*

Proof: Let R = $Z_{p^n}[i]$ be ring over Gaussian integer p≡2(mod4). If a+bi represent zero divisor then b+ai also gives zero divisor, and the vertices $(a_1+b_1i)\,\frac{1}{2}p^n\cdot(1+i) = 0$ and $(b_1+ia_1)\,\frac{1}{2}p^n\cdot(1+i) = 0$, i.e., some of rows of matrix due to above product will be identical so determinant of matrix is zero.

Theorem 4.2: *Let $Z_n[i]$ be ring of Gaussian integer under modulo 'n'. Consider $Z_{p^n}[i]$, p≡2(mod4), p be prime. Let M be adjacency matrix corresponding to zero divisor graph of $Z_{p^n}[i]$. Then trace of adjacency matrix is always natural number, k>1.*

Proof: Let R = $Z_{p^n}[i]$, let p≡2(mod4), i.e., form of 2^n in this case vertex $\frac{1}{2}p^n \cdot (1+i)$ represents zero divisor and in fact, when graph is formed vertex $\frac{1}{2}p^n \cdot (1+i)$ always have self loop, when adjacency matrix is constructed then $\frac{1}{2}p^n \cdot (1+i)$ vertex have entry 1(diagonal form), so trace is at least k, k>1.

5. Conclusion:

In this paper, we study adjacency matrices for zero divisor graph over finite rings of Gaussian integer. Graphs are the most ubiquitous models of both natural and human made structures. In computer science, zero divisor graphs are used to represent networks of communication, network flow, clique problems. Art gallery and museum guard problem are a well-studied visibility problem in computational geometry**.**

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